Error Analysis and Problem Condition

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Informatik 12:
Software and Tools for Computational Engineering

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Objective

- Introduction to error analysis and problem condition

Learning Outcomes

- You will understand
  - absolute and relative errors
  - absolute and relative condition of functions
  - link with first derivatives.
- You will be able to
  - perform error analysis
  - estimate the condition

  of simple functions.

Let $y = p \cdot x$ be evaluated at erroneous argument $x + \Delta x$, e.g, the value of x might be hard or even impossible to measure precisely.

The absolute error Δy scales as p since

$$y + \Delta y = p \cdot (x + \Delta x) = p \cdot x + p \cdot \Delta x$$

yields

$$\Delta y = p \cdot \Delta x \ .$$

The relative (w.r.t. the correct function value y) error becomes equal to

$$\delta y \equiv \frac{\Delta y}{y} = \frac{p \cdot \Delta x}{p \cdot x} = \frac{\Delta x}{x} \ .$$

It turns out to be independent of p in the linear case.

Errors

are relative

- distance
  - car navigation vs. brain surgery
- time
  - meeting people vs. high-frequency trading (of financial products)
- weight
  - your body vs. gold
- temperature
  - your coffee vs. earth's inner core

$\rightarrow$ Sensitivity

- of simulations (evaluations of models) wrt. errors
- of errors wrt. uncertain model parameters

- modelling / abstraction (impact to be minimized)
- measurements / observations (impact to be minimized)
- computer arithmetic (impact to be understood / minimized)
- reinforcement due to problem condition (to be understood / minimized)
- *bugs* (to be avoided / eliminated)
- mixtures of the above

Absolute Error Analysis

Example

Let $y = f(x) = x^2$ be evaluated at an erroneous point $\tilde{x} = x + \Delta x$. Consider the error in the result $\tilde{y} = f(\tilde{x})$.

The absolute result error Δy due to the absolute argument error Δx is equal to

$$\Delta y = f(x) - f(x + \Delta x)\;,$$

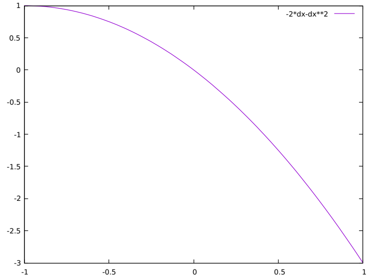
e.g,

- $x = 0,\ \Delta x = 0.1 \quad \Rightarrow \quad \Delta y = 0^2 - 0.1^2 = -0.01$
- $x = -0.5,\ \Delta x = 0.1 \quad \Rightarrow \quad \Delta y = (-0.5)^2 - 0.4^2 = 0.09$
- $x = 1,\ \Delta x = -0.1 \quad \Rightarrow \quad \Delta y = 1^2 - 0.9^2 = 0.19$
- $x = -2,\ \Delta x = -0.1 \quad \Rightarrow \quad \Delta y = (-2)^2 - (-2.1)^2 = -0.41$
- $x = 10,\ \Delta x = 0.1 \quad \Rightarrow \quad \Delta y = 10^2 - 10.1^2 = -2.01$

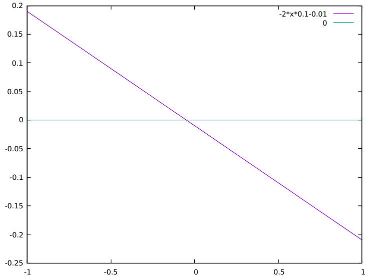
... growth of $|\Delta y|$

Absolute error analysis of $y = f(x) = x^2$ yields

$$\Delta y = f(x) - f(x + \Delta x) = x^2 - (x + \Delta x)^2 = -2 \cdot x \cdot \Delta x - \Delta x^2$$



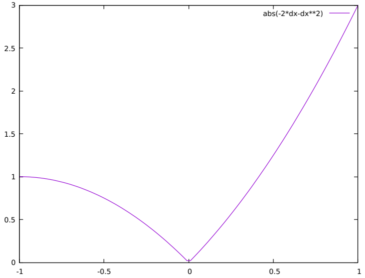
$\Delta y(x = 1, \Delta x)$



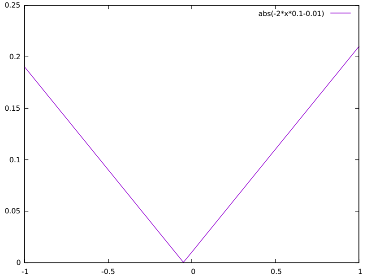
$\Delta y(x, \Delta x = 0.1)$

Typically, the absolute value $|\Delta y|$ is used to quantify the impact of an absolute error Δx in x on the result y, e.g,

$$|\Delta y| = |f(x) - f(x + \Delta x)| = |x^2 - (x + \Delta x)^2| = | - 2 \cdot x \cdot \Delta x - \Delta x^2|$$



$|\Delta y(x = 1, \Delta x)|$



$|\Delta y(x, \Delta x = 0.1)|$

The relative result error δy due to the absolute argument error Δx is equal to

$$\delta y = \frac{\Delta y}{y} = \frac{f(x) - f(x + \Delta x)}{f(x)} ,$$

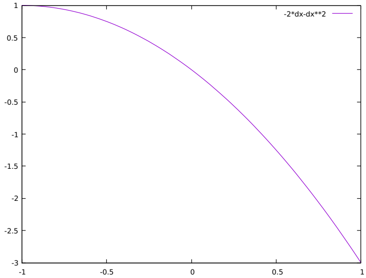
e.g,

- $x = 0, \; \Delta x = 0.1 \quad \Rightarrow \quad \delta y = \frac{0^2 - 0.1^2}{0^2} = -\infty$
- $x = -0.5, \; \Delta x = 0.1 \quad \Rightarrow \quad \delta y = \frac{(-0.5)^2 - 0.4^2}{(-0.5)^2} = 0.36$
- $x = 1, \; \Delta x = -0.1 \quad \Rightarrow \quad \delta y = \frac{1^2 - 0.9^2}{1^2} = 0.19$
- $x = -2, \; \Delta x = -0.1 \quad \Rightarrow \quad \delta y = \frac{(-2)^2 - (-2.1)^2}{(-2)^2} = -0.1025$
- $x = 10, \; \Delta x = 0.1 \quad \Rightarrow \quad \delta y = \frac{10^2 - 10.1^2}{10^2} = -0.0201$

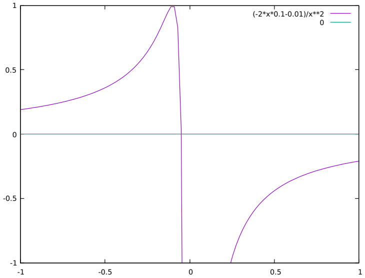
... shrinkage of $|\delta y|$

Relative error analysis of $y = f(x) = x^2$ yields

$$\delta y = \frac{f(x) - f(x + \Delta x)}{f(x)} = \frac{x^2 - (x + \Delta x)^2}{x^2} = \frac{-2 \cdot x \cdot \Delta x - \Delta x^2}{x^2}$$



$\delta y(x = 1, \Delta x)$



$\delta y(x, \Delta x = 0.1)$

Rule of de l'Hospital

Evaluating Limits of Differentiable Functions

Let $f, g : \boldsymbol{R} \to \boldsymbol{R}$ be differentiable in a neighborhood of $\tilde{x} \in \boldsymbol{R}$ and

$$\lim_{x \to \tilde{x}} f(x) = \lim_{x \to \tilde{x}} g(x) = 0$$

or

$$\lim_{x \to \tilde{x}} f(x) = \pm\infty \quad \text{and} \quad \lim_{x \to \tilde{x}} g(x) = \pm\infty \; .$$

Then

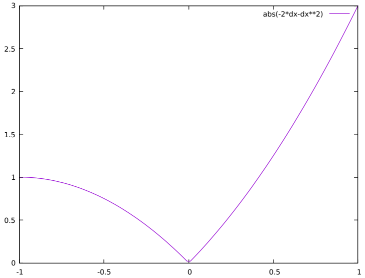
$$\lim_{x \to \tilde{x}} \frac{f(x)}{g(x)} = \lim_{x \to \tilde{x}} \frac{f'(x)}{g'(x)} \; .$$

For example,

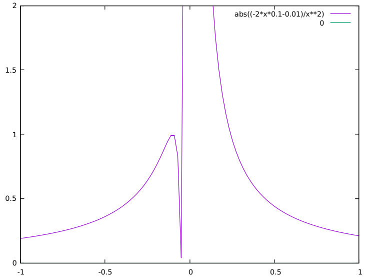
$$\lim_{x \to \pm\infty} \frac{-2 \cdot x \cdot \Delta x - \Delta x^2}{x^2} = \lim_{x \to \pm\infty} \frac{-2 \cdot \Delta x}{2 \cdot x} = \lim_{x \to \pm\infty} \frac{-\Delta x}{x} = 0$$

Typically, the absolute value $|\delta y|$ is considered, e.g,

$$|\delta y| = \left| \frac{-2 \cdot x \cdot \Delta x - \Delta x^2}{x^2} \right| = \frac{|-2 \cdot x \cdot \Delta x - \Delta x^2|}{|x^2|}$$



$|\delta y(x = 1, \Delta x)|$



$|\delta y(x, \Delta x = 0.1)|$

The effect of an error $0 < |\Delta x| < 1$ in the argument of a function $y = f(x)$ is of fundamental interest for numerical simulation. Strong amplification of this error implies rigorous requirements for the accuracy of the input.

Investigation of the asymptotic $(\Delta x \to 0)$ behavior of

$$\left|\frac{\Delta y}{\Delta x}\right| = \frac{|\Delta y|}{|\Delta x|} .$$

yields the absolute condition of f at x defined as

$$\mathcal{K}_f(x) \equiv \lim_{\Delta x \to 0} \frac{|\Delta y|}{|\Delta x|} .$$

For $y = x^2$ we observe

- $x = 0,\ \ \Delta x = 0.1 \quad \Rightarrow \quad \frac{|0^2 - 0.1^2|}{|0.1|} = |-0.1| = 0.1$
- $x = -0.5,\ \ \Delta x = 0.1 \quad \Rightarrow \quad \frac{|(-0.5)^2 - (-0.4)^2|}{|0.1|} = |0.9| = 0.9$
- $x = 1,\ \ \Delta x = -0.1 \quad \Rightarrow \quad \frac{|1^2 - 0.9^2|}{|-0.1|} = |-1.9| = 1.9$

and

$$\mathcal{K}_f(x) \equiv \lim_{\Delta x \to 0} \frac{|\Delta y|}{|\Delta x|} = \lim_{\Delta x \to 0} \frac{|-2 \cdot x \cdot \Delta x + \Delta x^2|}{|\Delta x|}$$

$$= \lim_{\Delta x \to 0} \left| \frac{-2 \cdot x \cdot \Delta x + \Delta x^2}{\Delta x} \right| = \left| \lim_{\Delta x \to 0} \frac{-2 \cdot x \cdot \Delta x + \Delta x^2}{\Delta x} \right|$$

$$= \left| \lim_{\Delta x \to 0} \frac{-2 \cdot x + 2 \cdot \Delta x}{1} \right| = |-2 \cdot x|$$

Arguments similar to the ones made for absolute vs. relative errors suggest that relative condition should be considered instead of absolute condition.

Investigation of the asymptotic ($\Delta x \to 0$) behavior of

$$\left|\frac{\delta y}{\delta x}\right| = \left|\frac{\Delta y}{\Delta x} \cdot \frac{x}{y}\right| = \frac{|\Delta y \cdot x|}{|\Delta x \cdot y|}$$

yields the [relative] condition of f at x as

$$\kappa_f(x) \equiv \lim_{\Delta x \to 0} \left|\frac{\delta y}{\delta x}\right| = \lim_{\Delta x \to 0} \left|\frac{\Delta y}{\Delta x} \cdot \frac{x}{y}\right|$$

Condition of a Function

[Relative] Condition

For $y = f(x) = x^2$ we observe

- $x = 0,\ \Delta x = 0.1 \quad \Rightarrow \quad \left| \frac{0^2 - 0.1^2}{0.1} \cdot \frac{0}{0} \right| = \text{NaN}$
- $x = -0.5,\ \Delta x = 0.1 \quad \Rightarrow \quad \left| \frac{(-0.5)^2 - (-0.4)^2}{0.1} \cdot \frac{-0.5}{0.25} \right| = |0.9 \cdot (-2)| = 1.8$
- $x = 1,\ \Delta x = -0.1 \quad \Rightarrow \quad \left| \frac{1^2 - 0.9^2}{-0.1} \cdot \frac{1}{1} \right| = |-1.9 \cdot 1| = 1.9$

and

$$
\begin{aligned}
\kappa_f(x) &\equiv \lim_{\Delta x \to 0} \left| \frac{\delta y}{\delta x} \right| = \lim_{\Delta x \to 0} \left| \frac{\Delta y}{\Delta x} \cdot \frac{x}{y} \right| \\
&= \lim_{\Delta x \to 0} \left| \frac{-2 \cdot x \cdot \Delta x + \Delta x^2}{\Delta x} \cdot \frac{x}{x^2} \right| = \lim_{\Delta x \to 0} \left| -2 - \frac{2 \cdot \Delta x}{x} \right| = 2
\end{aligned}
$$

Absolute Condition

Link with First Derivative (Linearization)

For $f(x) = x^2$

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x) - f(x+\Delta x)}{\Delta x} =_{\text{de l'Hospital}} \lim_{\Delta x \to 0} \frac{\frac{d}{d\Delta x}(f(x) - f(x+\Delta x))}{\frac{d}{d\Delta x}\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{d}{d\Delta x}(-f(x+\Delta x))$$

$$= \lim_{\Delta x \to 0} \frac{d}{d\Delta x}\left(-(x+\Delta x)^2\right)$$

$$= \lim_{\Delta x \to 0} \frac{d}{d\Delta x}\left(-(x^2 + 2\cdot x\cdot \Delta x + \Delta x^2)\right)$$

$$= \lim_{\Delta x \to 0} -2\cdot x - 2\cdot \Delta x = -2\cdot x = -\frac{d}{dx}f(x)$$

$$\mathcal{K}_f(x) = \lim_{\Delta x \to 0} \left|\frac{\Delta y}{\Delta x}\right| = \left|\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}\right| = \left|\frac{d}{dx}f(x)\right|$$

[Relative] Condition

Link with First Derivative (Linearization)

$$\lim_{\Delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\Delta x \to 0} \frac{f(x) - f(x+\Delta x)}{\Delta x} \cdot \frac{x}{f(x)}$$

$$=_{\text{de l'Hospital}} \lim_{\Delta x \to 0} \frac{x}{f(x)} \cdot \frac{\frac{d}{d\Delta x}(f(x) - f(x+\Delta x))}{\frac{d}{d\Delta x}\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{x}{f(x)} \cdot \frac{d}{d\Delta x}(-f(x+\Delta x))$$

$$= \lim_{\Delta x \to 0} \frac{x}{x^2} \cdot \frac{d}{d\Delta x}(-(x+\Delta x)^2)$$

$$= \lim_{\Delta x \to 0} \frac{1}{x} \cdot \frac{d}{d\Delta x}(-(x^2 + 2 \cdot x \cdot \Delta x + \Delta x^2))$$

$$= \lim_{\Delta x \to 0} \frac{1}{x} \cdot (-2 \cdot x - 2 \cdot \Delta x) = -2 = -\frac{x}{f(x)} \cdot \frac{d}{dx} f(x)$$

$$\kappa_f(x) = \lim_{\Delta x \to 0} \left| \frac{x}{f(x)} \cdot \frac{\Delta y}{\Delta x} \right| = \left| \lim_{\Delta x \to 0} \frac{x}{f(x)} \cdot \frac{\Delta y}{\Delta x} \right| = \left| \frac{x}{f(x)} \cdot \frac{d}{dx} f(x) \right|$$

The Mean Value Theorem is an extension of the Intermediate Value Theorem to first derivatives.

Let $y = f(x)$ be continuously differentiable in a neighborhood of x covering potential input errors Δx. Within this neighborhood there exists a $\tilde{x}$ such that

$$|f(x) - f(x + \Delta x)| = f'(\tilde{x}) \cdot |\Delta x| \, .$$

There is a tangent within the interval which is parallel to the secant defined by the endpoints.

Typically, the value of $\tilde{x}$ is unknown.

Let $y = f(x)$ be continuously differentiable in a neighborhood N of x covering potential input errors Δx. According to the mean value theorem there is some point $\tilde{x}$ within this neighborhood for which

$$|\Delta y| \equiv |f(x) - f(x + \Delta x)| = f'(\tilde{x}) \cdot |\Delta x| \ .$$

To bound the error we consider the worst case, that is

$$|\Delta y| \leq \max_{\tilde{x} \in N} |f'(\tilde{x})| \cdot |\Delta x| \ .$$

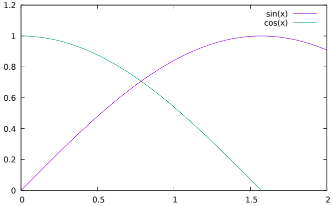
An upper bound for the right-hand side can be computed in interval arithmetic. Approximations for this error can be estimated as

$$|\Delta y| \approx |f'(\tilde{x})| \cdot |\Delta x|$$

for arbitrary $\tilde{x} \in N$, e.g, $\tilde{x} = x + \Delta x$.

Consider $y = \sin(x)$ at $x = 1$ and $\Delta x = 0.1$.

$$\begin{aligned}|\Delta y| &\leq \max_{\tilde{x} \in N} |f'(\tilde{x})| \cdot |\Delta x| \\ &= \max_{\tilde{x} \in [1,1.1]} |\cos(\tilde{x})| \cdot 0.1 \\ &= |\cos(1)| \cdot 0.1 \\ &\approx 0.54 \cdot 0.1 \\ &= 0.054\end{aligned}$$



$$|\delta y| \leq \frac{|\Delta y|}{|y|} \approx \frac{0.054}{0.84} = 0.064$$

Summary

- absolute vs. relative errors
- absolute vs. relative condition
- mean value theorem

Next Steps

- Practice error analysis on more complex functions.
- Continue the course to find out more ...