Error Analysis and Problem Condition

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Objective

Introduction to error analysis and problem condition

Learning Outcomes

You will understand
  - absolute and relative errors
  - absolute and relative condition of functions
  - link with first derivatives.

You will be able to
  - perform error analysis
  - estimate the condition of simple functions.
Errors
Absolute and Relative Errors

Let $y = p \cdot x$ be evaluated at erroneous argument $x + \Delta x$, e.g., the value of $x$ might be hard or even impossible to measure precisely.

The absolute error $\Delta y$ scales as $p$ since

$$y + \Delta y = p \cdot (x + \Delta x) = p \cdot x + p \cdot \Delta x$$

yields

$$\Delta y = p \cdot \Delta x.$$ 

The relative (w.r.t. the correct function value $y$) error becomes equal to

$$\delta y \equiv \frac{\Delta y}{y} = \frac{p \cdot \Delta x}{p \cdot x} = \frac{\Delta x}{x}.$$ 

It turns out to be independent of $p$ in the linear case.
Errors are relative

- distance
  - car navigation vs. brain surgery
- time
  - meeting people vs. high-frequency trading (of financial products)
- weight
  - your body vs. gold
- temperature
  - your coffee vs. earth’s inner core

→ Sensitivity
- of simulations (evaluations of models) wrt. errors
- of errors wrt. uncertain model parameters
Errors

Sources

▶ modelling / abstraction (impact to be minimized)
▶ measurements / observations (impact to be minimized)
▶ computer arithmetic (impact to be understood / minimized)
▶ reinforcement due to problem condition (to be understood / minimized)
▶ bugs (to be avoided / eliminated)
▶ mixtures of the above
Absolute Error Analysis

Example

Let \( y = f(x) = x^2 \) be evaluated at an erroneous point \( \tilde{x} = x + \Delta x \). Consider the error in the result \( \tilde{y} = f(\tilde{x}) \).

The absolute result error \( \Delta y \) due to the absolute argument error \( \Delta x \) is equal to

\[
\Delta y = f(x) - f(x + \Delta x),
\]

e.g,

\[
\begin{align*}
\text{\( x = 0 \), \( \Delta x = 0.1 \) } & \quad \Rightarrow \quad \Delta y = 0^2 - 0.1^2 = -0.01 \\
\text{\( x = -0.5 \), \( \Delta x = 0.1 \) } & \quad \Rightarrow \quad \Delta y = (-0.5)^2 - 0.4^2 = 0.09 \\
\text{\( x = 1 \), \( \Delta x = -0.1 \) } & \quad \Rightarrow \quad \Delta y = 1^2 - 0.9^2 = 0.19 \\
\text{\( x = -2 \), \( \Delta x = -0.1 \) } & \quad \Rightarrow \quad \Delta y = (-2)^2 - (-2.1)^2 = -0.41 \\
\text{\( x = 10 \), \( \Delta x = 0.1 \) } & \quad \Rightarrow \quad \Delta y = 10^2 - 10.1^2 = -2.01
\end{align*}
\]

... growth of \(|\Delta y|\)
Absolute error analysis of $y = f(x) = x^2$ yields

$$\Delta y = f(x) - f(x + \Delta x) = x^2 - (x + \Delta x)^2 = -2 \cdot x \cdot \Delta x - \Delta x^2$$

$\Delta y(x = 1, \Delta x)$

$\Delta y(x, \Delta x = 0.1)$
Typically, the absolute value $|\Delta y|$ is used to quantify the impact of an absolute error $\Delta x$ in $x$ on the result $y$, e.g,

$$|\Delta y| = |f(x) - f(x + \Delta x)| = |x^2 - (x + \Delta x)^2| = |-2 \cdot x \cdot \Delta x - \Delta x^2|$$

$$|\Delta y(x = 1, \Delta x)|$$  $$|\Delta y(x, \Delta x = 0.1)|$$
Relative Error Analysis

Example

The relative result error $\delta y$ due to the absolute argument error $\Delta x$ is equal to

$$\delta y = \frac{\Delta y}{y} = \frac{f(x) - f(x + \Delta x)}{f(x)},$$

e.g,

$\blacktriangleright$ $x = 0, \Delta x = 0.1 \Rightarrow \delta y = \frac{0^2 - 0.1^2}{0^2} = -\infty$

$\blacktriangleright$ $x = -0.5, \Delta x = 0.1 \Rightarrow \delta y = \frac{(-0.5)^2 - 0.4^2}{(-0.5)^2} = 0.36$

$\blacktriangleright$ $x = 1, \Delta x = -0.1 \Rightarrow \delta y = \frac{1^2 - 0.9^2}{1^2} = 0.19$

$\blacktriangleright$ $x = -2, \Delta x = -0.1 \Rightarrow \delta y = \frac{(-2)^2 - (-2.1)^2}{(-2)^2} = -0.1025$

$\blacktriangleright$ $x = 10, \Delta x = 0.1 \Rightarrow \delta y = \frac{10^2 - 10.1^2}{10^2} = -0.0201$

... shrinkage of $|\delta y|$
Relative error analysis of \( y = f(x) = x^2 \) yields

\[
\delta y = \frac{f(x) - f(x + \Delta x)}{f(x)} = \frac{x^2 - (x + \Delta x)^2}{x^2} = \frac{-2 \cdot x \cdot \Delta x - \Delta x^2}{x^2}
\]

\[
\delta y(x = 1, \Delta x)
\]

\[
\delta y(x, \Delta x = 0.1)
\]
Rule of de l’Hospital
Evaluating Limits of Differentiable Functions

Let \( f, g : \mathbb{R} \to \mathbb{R} \) be differentiable in a neighborhood of \( \tilde{x} \in \mathbb{R} \) and

\[
\lim_{x \to \tilde{x}} f(x) = \lim_{x \to \tilde{x}} g(x) = 0
\]

or

\[
\lim_{x \to \tilde{x}} f(x) = \pm \infty \quad \text{and} \quad \lim_{x \to \tilde{x}} g(x) = \pm \infty.
\]

Then

\[
\lim_{x \to \tilde{x}} \frac{f(x)}{g(x)} = \lim_{x \to \tilde{x}} \frac{f'(x)}{g'(x)}.
\]

For example,

\[
\lim_{x \to \pm \infty} \frac{-2 \cdot x \cdot \Delta x - \Delta x^2}{x^2} = \lim_{x \to \pm \infty} \frac{-2 \cdot \Delta x}{2 \cdot x} = \lim_{x \to \pm \infty} \frac{-\Delta x}{x} = 0
\]
Relative Error Analysis

Example

Typically, the absolute value $|\delta y|$ is considered, e.g,

$$|\delta y| = \left| \frac{-2 \cdot x \cdot \Delta x - \Delta x^2}{x^2} \right| = \frac{|-2 \cdot x \cdot \Delta x - \Delta x^2|}{|x^2|}$$

| $\delta y(x = 1, \Delta x)$ | $\delta y(x, \Delta x = 0.1)$ |
The effect of an error $0 < |\Delta x| < 1$ in the argument of a function $y = f(x)$ is of fundamental interest for numerical simulation. Strong amplification of this error implies rigorous requirements for the accuracy of the input.

Investigation of the asymptotic ($\Delta x \to 0$) behavior of

$$\left| \frac{\Delta y}{\Delta x} \right| = \frac{|\Delta y|}{|\Delta x|}.$$  

yields the **absolute condition** of $f$ at $x$ defined as

$$k_f(x) \equiv \lim_{\Delta x \to 0} \frac{|\Delta y|}{|\Delta x|}.$$
Absolute Condition of a Function

Example

For \( y = x^2 \) we observe

\[ x = 0, \ \Delta x = 0.1 \Rightarrow \left| \frac{0^2 - 0.1^2}{0.1} \right| = | - 0.1 | = 0.1 \]

\[ x = -0.5, \ \Delta x = 0.1 \Rightarrow \left| \frac{(-0.5)^2 - (-0.4)^2}{0.1} \right| = |0.9| = 0.9 \]

\[ x = 1, \ \Delta x = -0.1 \Rightarrow \left| \frac{1^2 - 0.9^2}{-0.1} \right| = | - 1.9 | = 1.9 \]

and

\[ K_f(x) \equiv \lim_{\Delta x \to 0} \frac{|\Delta y|}{|\Delta x|} = \lim_{\Delta x \to 0} \left| \frac{-2 \cdot x \cdot \Delta x + \Delta x^2}{|\Delta x|} \right| = \left| \lim_{\Delta x \to 0} \frac{-2 \cdot x \cdot \Delta x + \Delta x^2}{\Delta x} \right| = \left| \lim_{\Delta x \to 0} \frac{-2 \cdot x + 2 \cdot \Delta x}{1} \right| = | - 2 \cdot x | \]
Arguments similar to the ones made for absolute vs. relative errors suggest that relative condition should be considered instead of absolute condition.

Investigation of the asymptotic ($\Delta x \to 0$) behavior of

$$\left| \frac{\delta y}{\delta x} \right| = \left| \frac{\Delta y}{\Delta x} \cdot \frac{x}{y} \right| = \frac{|\Delta y \cdot x|}{|\Delta x \cdot y|}$$

yields the [relative] condition of $f$ at $x$ as

$$\kappa_f(x) \equiv \lim_{\Delta x \to 0} \left| \frac{\delta y}{\delta x} \right| = \lim_{\Delta x \to 0} \left| \frac{\Delta y}{\Delta x} \cdot \frac{x}{y} \right|$$
Condition of a Function

[Relative] Condition

For \( y = f(x) = x^2 \) we observe

\[ x = 0, \ \Delta x = 0.1 \quad \Rightarrow \quad \left| \frac{0^2-0.1^2}{0.1} \cdot \frac{0}{0} \right| = \text{NaN} \]

\[ x = -0.5, \ \Delta x = 0.1 \quad \Rightarrow \quad \left| \frac{(-0.5)^2-(-0.4)^2}{0.1} \cdot \frac{-0.5}{0.25} \right| = |0.9 \cdot (-2)| = 1.8 \]

\[ x = 1, \ \Delta x = -0.1 \quad \Rightarrow \quad \left| \frac{1^2-0.9^2}{-0.1} \cdot \frac{1}{1} \right| = | -1.9 \cdot 1| = 1.9 \]

and

\[
\kappa_f(x) \equiv \lim_{\Delta x \to 0} \left| \frac{\delta y}{\delta x} \right| = \lim_{\Delta x \to 0} \left| \frac{\Delta y}{\Delta x} \cdot \frac{x}{y} \right| \\
= \lim_{\Delta x \to 0} \left| \frac{-2 \cdot x \cdot \Delta x + \Delta x^2}{\Delta x} \cdot \frac{x}{x^2} \right| = \lim_{\Delta x \to 0} \left| -2 - \frac{2 \cdot \Delta x}{x} \right| = 2
\]
Absolute Condition
Link with First Derivative (Linearization)

For \( f(x) = x^2 \)

\[
\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x) - f(x + \Delta x)}{\Delta x} = \text{de l'Hospital} \lim_{\Delta x \to 0} \frac{\frac{d}{d\Delta x} (f(x) - f(x + \Delta x))}{\frac{d}{d\Delta x} \Delta x}
\]

\[
= \lim_{\Delta x \to 0} \frac{d}{d\Delta x} (-f(x + \Delta x))
\]

\[
= \lim_{\Delta x \to 0} \frac{d}{d\Delta x} (- (x + \Delta x)^2)
\]

\[
= \lim_{\Delta x \to 0} \frac{d}{d\Delta x} (- (x^2 + 2 \cdot x \cdot \Delta x + \Delta x^2))
\]

\[
= \lim_{\Delta x \to 0} -2 \cdot x - 2 \cdot \Delta x = -2 \cdot x = -\frac{d}{dx} f(x)
\]

\[
\mathcal{K}_f(x) = \lim_{\Delta x \to 0} \left| \frac{\Delta y}{\Delta x} \right| = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \left| \frac{d}{dx} f(x) \right|
\]
[Relative] Condition
Link with First Derivative (Linearization)

\[
\lim_{\Delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\Delta x \to 0} \frac{f(x) - f(x + \Delta x)}{\Delta x} \cdot \frac{x}{f(x)}
\]

\[= \text{de l'Hospital} \lim_{\Delta x \to 0} \frac{x}{f(x)} \cdot \frac{d}{d\Delta x} \left( f(x) - f(x + \Delta x) \right) \cdot \frac{d}{d\Delta x} \Delta x \]

\[= \lim_{\Delta x \to 0} \frac{x}{f(x)} \cdot \frac{d}{d\Delta x} \left( -f(x + \Delta x) \right) \]

\[= \lim_{\Delta x \to 0} \frac{x}{x^2} \cdot \frac{d}{d\Delta x} \left( -(x + \Delta x)^2 \right) \]

\[= \lim_{\Delta x \to 0} \frac{1}{x} \cdot \frac{d}{d\Delta x} \left( -(x^2 + 2 \cdot x \cdot \Delta x + \Delta x^2) \right) \]

\[= \lim_{\Delta x \to 0} \frac{1}{x} \cdot (-2 \cdot x - 2 \cdot \Delta x) = -2 = -\frac{x}{f(x)} \cdot \frac{d}{dx} f(x) \]

\[\kappa_f(x) = \lim_{\Delta x \to 0} \left| \frac{x}{f(x)} \cdot \frac{\Delta y}{\Delta x} \right| = \left| \lim_{\Delta x \to 0} \frac{x}{f(x)} \cdot \frac{\Delta y}{\Delta x} \right| = \left| \frac{x}{f(x)} \cdot \frac{d}{dx} f(x) \right| \]

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The **Mean Value Theorem** is an extension of the Intermediate Value Theorem to first derivatives.

Let \( y = f(x) \) be continuously differentiable in a neighborhood of \( x \) covering potential input errors \( \Delta x \). Within this neighborhood there exists a \( \tilde{x} \) such that

\[
|f(x) - f(x + \Delta x)| = f'(\tilde{x}) \cdot |\Delta x|.
\]

There is a tangent within the interval which is parallel to the secant defined by the endpoints.

Typically, the value of \( \tilde{x} \) is unknown.
Mean Value Theorem

Application the Error Analysis

Let \( y = f(x) \) be continuously differentiable in a neighborhood \( N \) of \( x \) covering potential input errors \( \Delta x \). According to the mean value theorem there is some point \( \tilde{x} \) within this neighborhood for which

\[
|\Delta y| \equiv |f(x) - f(x + \Delta x)| = f'(\tilde{x}) \cdot |\Delta x|.
\]

To bound the error we consider the worst case, that is

\[
|\Delta y| \leq \max_{\tilde{x} \in N} |f'(\tilde{x})| \cdot |\Delta x|.
\]

An upper bound for the right-hand side can be computed in interval arithmetic. Approximations for this error can be estimated as

\[
|\Delta y| \approx |f'(\tilde{x})| \cdot |\Delta x|
\]

for arbitrary \( \tilde{x} \in N \), e.g, \( \tilde{x} = x + \Delta x \).
Consider $y = \sin(x)$ at $x = 1$ and $\Delta x = 0.1$.

$$|\Delta y| \leq \max_{\tilde{x} \in N} |f'(\tilde{x})| \cdot |\Delta x|$$

$$= \max_{\tilde{x} \in [1,1.1]} |\cos(\tilde{x})| \cdot 0.1$$

$$= |\cos(1)| \cdot 0.1$$

$$\approx 0.54 \cdot 0.1$$

$$= 0.054$$

$$|\delta y| \leq \frac{|\Delta y|}{|y|} \approx \frac{0.054}{0.84} = 0.064$$
Summary

- absolute vs. relative errors
- absolute vs. relative condition
- mean value theorem

Next Steps

- Practice error analysis on more complex functions.
- Continue the course to find out more ...