Modern Family

Convex Minimization in *R*

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Informatik 12:
Software and Tools for Computational Engineering

RWTH Aachen University

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Objective and Learning Outcomes

Least-Squares Objective

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- Implementation

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Outline

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Summary and Next Steps

Objective

- Introduction to sample code approaching the Modern Family example as a general convex minimization problem.

Learning Outcomes

- You will understand
  - the formulation of the Modern Family example as a general convex minimization problem
  - its solution using the general-purpose optimization methods bisection, steepest descent, Newton's method.
- You will be able to
  - run the sample code
  - compare the results

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Summary and Next Steps

We state the Modern Family example for model $y = f(p, x)$ as minimization of the error function

$$E(p, \mathbf{x}, \mathbf{y}) = \sum_{i=0}^{m-1} (f(p, x_i) - y_i)^2$$

Linear (in p)

$$y = p \cdot x$$

and nonlinear

$$y = (p \cdot x)^2$$

models are considered.

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Necessary

$$\frac{dE}{dp}(p, \mathbf{x}, \mathbf{y}) = 2 \cdot \sum_{i=0}^{m-1} \left((f(p, x_i) - y_i) \cdot \frac{df}{dp}(p, x_i) \right) \quad \to 0$$

Sufficient

$$\frac{d^2E}{dp^2}(p, \mathbf{x}, \mathbf{y}) = 2 \cdot \sum_{i=0}^{m-1} \left(\left(\frac{df}{dp}(p, x_i) \right)^2 + (f(p, x_i) - y_i) \cdot \frac{d^2f}{dp^2}(p, x_i) \right) \quad > 0$$

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For $y = f(p, x) = p \cdot x$

$$E(p, \mathbf{x}, \mathbf{y}) = \sum_{i=0}^{m-1} (f(p, x_i) - y_i)^2 = \sum_{i=0}^{m-1} (x_i \cdot p - y_i)^2 = \|p \cdot \mathbf{x} - \mathbf{y}\|_2^2$$

and hence

$$\frac{dE}{dp}(p, \mathbf{x}, \mathbf{y}) = 2 \cdot \sum_{i=0}^{m-1} (x_i \cdot p - y_i) \cdot x_i \quad \rightarrow 0$$

and

$$\frac{d^2 E}{dp^2}(p, \mathbf{x}, \mathbf{y}) = 2 \cdot \sum_{i=0}^{m-1} x_i^2 \quad > 0!$$

We present implementations for the solution of the nonlinear equation

$$2 \cdot \sum_{i=0}^{m-1} (x_i \cdot p - y_i) \cdot x_i = 0$$

using

- bisection
- steepest descent
- Newton's method.

See source code.

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For $y = f(p, x) = (p \cdot x)^2$

$$E(p, \mathbf{x}, \mathbf{y}) = \sum_{i=0}^{m-1} (f(p, x_i) - y_i)^2 = \sum_{i=0}^{m-1} \left((p \cdot x_i)^2 - y_i\right)^2$$

and hence

$$\frac{dE}{dp}(p, \mathbf{x}, \mathbf{y}) = 4 \cdot p \cdot \sum_{i=0}^{m-1} p^2 \cdot x_i^4 - x_i^2 \cdot y_i \quad \rightarrow 0$$

and

$$\frac{d^2E}{dp^2}(p, \mathbf{x}, \mathbf{y}) = 4 \cdot \sum_{i=0}^{m-1} 3 \cdot p^2 \cdot x_i^4 - x_i^2 \cdot y_i \quad > 0!$$

We present implementations for the solution of the nonlinear equation

$$4 \cdot p \cdot \sum_{i=0}^{m-1} p^2 \cdot x_i^4 - x_i^2 \cdot y_i = 0$$

using

- bisection
- steepest descent
- Newton's method with differentiation performed
  - symbolically
  - approximately (finite differences)
  - algorithmically (dco/c++)

See source code.

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Summary and Next Steps

Summary

- Introduction to sample code approaching the Modern Family example as a general convex minimization problem.

Next Steps

- Run the sample code.
- Compare the results.
- Continue the course to find out more ...