Modern Family

Convex Minimization in $\boldsymbol{R}^n$

Uwe Naumann

Informatik 12:
Software and Tools for Computational Engineering (STCE)

RWTH Aachen University

Contents

Outline

Objective

- Introduction to sample code approaching the Modern Family example as a general convex minimization problem.

Learning Outcomes

- You will understand
  - the formulation of the Modern Family example as a general convex minimization problem
  - its solution using the general-purpose optimization methods steepest descent and Newton's method.
- You will be able to
  - run the sample code
  - compare the results

Outline

We state the Modern Family example for model

$$y = f(\mathbf{p}, \mathbf{x}) : \boldsymbol{R}^n \times \boldsymbol{R}^n \rightarrow \boldsymbol{R}$$

as minimization of the error function

$$E(\mathbf{p}, X, \mathbf{y}) = \sum_{i=0}^{m-1} (f(\mathbf{p}, \mathbf{x}_i^T) - y_i)^2$$

Linear (in **p**)

$$y = \mathbf{p}^T \cdot \mathbf{x}$$

and nonlinear

$$y = (\mathbf{p}^T \cdot \mathbf{x})^2$$

models are considered.

Outline

Necessary

$$\frac{dE}{d\mathbf{p}}(\mathbf{p}, X, \mathbf{y}) = 2 \cdot \sum_{i=0}^{m-1} \left((f(\mathbf{p}, \mathbf{x}_i^T) - y_i) \cdot \frac{df}{d\mathbf{p}}(\mathbf{p}, \mathbf{x}_i^T) \right) \quad \rightarrow 0$$

Sufficient

$$\mathbf{v}^T \cdot \frac{d^2E}{d\mathbf{p}^2}(\mathbf{p}, X, \mathbf{y}) \cdot \mathbf{v} > 0 \quad \forall \mathbf{v} \neq \mathbf{0} \in R^n$$

where

$$\frac{d^2E}{d\mathbf{p}^2} = 2 \cdot \sum_{i=0}^{m-1} \left(\frac{df}{d\mathbf{p}}(\mathbf{p}, \mathbf{x}_i^T)^T \cdot \frac{df}{d\mathbf{p}}(\mathbf{p}, \mathbf{x}_i^T) + \frac{d^2f}{d\mathbf{p}^2}(\mathbf{p}, \mathbf{x}_i^T) \cdot (f(\mathbf{p}, \mathbf{x}_i^T) - y_i) \right)$$

Outline

For $y = f(\mathbf{p}, \mathbf{x}) = \mathbf{p}^T \cdot \mathbf{x} = \mathbf{x}^T \cdot \mathbf{p}$

$$E(\mathbf{p}, X, \mathbf{y}) = \sum_{i=0}^{m-1} (f(\mathbf{p}, \mathbf{x}_i^T) - y_i)^2 = \sum_{i=0}^{m-1} (\mathbf{x}_i \cdot \mathbf{p} - y_i)^2$$

and hence

$$\frac{dE}{d\mathbf{p}}(\mathbf{p}, X, \mathbf{y}) = 2 \cdot \sum_{i=0}^{m-1} (\mathbf{x}_i \cdot \mathbf{p} - y_i) \cdot \mathbf{x}_i \in \boldsymbol{R}^{1 \times n}$$

and

$$\frac{d^2 E}{d\mathbf{p}^2}(\mathbf{p}, X, \mathbf{y}) = 2 \cdot \sum_{i=0}^{m-1} \mathbf{x}_i^T \cdot \mathbf{x}_i \in \boldsymbol{R}^{n \times n} \ .$$

We present implementations for the solution of the convex unconstrained minimization problem

$$\min_{\mathbf{p} \in R^n} E(\mathbf{p}, X, \mathbf{y})$$

using

- steepest descent
- Newton's method.

See source code.

Outline

For $y = f(\mathbf{p},\mathbf{x}) = (\mathbf{p}^T \cdot \mathbf{x})^2 = (\mathbf{x}^T \cdot \mathbf{p})^2$

$$E(\mathbf{p}, X, \mathbf{y}) = \sum_{i=0}^{m-1} (f(\mathbf{p}, \mathbf{x}_i^T) - y_i)^2 = \sum_{i=0}^{m-1} (\mathbf{x}_i \cdot \mathbf{p} - y_i)^2$$

and hence

$$\frac{dE}{d\mathbf{p}}(\mathbf{p}, X, \mathbf{y}) = 4 \cdot \sum_{i=0}^{m-1} \left((\mathbf{x}_i \cdot \mathbf{p})^3 - y_i \cdot \mathbf{x}_i \cdot \mathbf{p}\right) \cdot \mathbf{x}_i$$

and

$$\frac{d^2E}{d\mathbf{p}^2}(\mathbf{p}, X, \mathbf{y}) = 4 \cdot \sum_{i=0}^{m-1} (3 \cdot (\mathbf{x}_i \cdot \mathbf{p})^2 - y_i) \cdot \mathbf{x}_i^T \cdot \mathbf{x}_i \ .$$

We present implementations for the solution of the convex unconstrained minimization problem

$$\min_{\mathbf{p} \in R^n} E(\mathbf{p}, X, \mathbf{y})$$

using

- steepest descent
- Newton's method with differentiation performed
  - symbolically
  - approximately (finite differences)
  - algorithmically (dco/c++)

See source code.

Outline

Summary

- Introduction to sample code approaching the Modern Family example as a general convex minimization problem.
- Solution of Modern Family problem using steepest descent and Newton's method.

Next Steps

- Run the sample code.
- Compare the results.
- Continue the course to find out more ...