Modern Family

Convex Minimization in $\mathbb{R}^n$

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Informatik 12:
Software and Tools for Computational Engineering (STCE)
RWTH Aachen University
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Summary and Next Steps
Objective

► Introduction to sample code approaching the Modern Family example as a general convex minimization problem.

Learning Outcomes

► You will understand
  ► the formulation of the Modern Family example as a general convex minimization problem
  ► its solution using the general-purpose optimization methods steepest descent and Newton’s method.

► You will be able to
  ► run the sample code
  ► compare the results
Outline

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Summary and Next Steps
We state the Modern Family example for model

\[ y = f(p, x) : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R} \]

as minimization of the error function

\[ E(p, X, y) = \sum_{i=0}^{m-1} (f(p, x_i^T) - y_i)^2 \]

Linear (in \( p \))

\[ y = p^T \cdot x \]

and nonlinear

\[ y = (p^T \cdot x)^2 \]

models are considered.
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Optimality Conditions

Necessary

$$\frac{dE}{dp}(p, X, y) = 2 \cdot \sum_{i=0}^{m-1} \left( (f(p, x_i^T) - y_i) \cdot \frac{df}{dp}(p, x_i^T) \right) \rightarrow 0$$

Sufficient

$$v^T \cdot \frac{d^2E}{dp^2}(p, X, y) \cdot v > 0 \quad \forall v \neq 0 \in \mathbb{R}^n$$

where

$$\frac{d^2E}{dp^2} = 2 \cdot \sum_{i=0}^{m-1} \left( \frac{df}{dp}(p, x_i^T)^T \cdot \frac{df}{dp}(p, x_i^T) + \frac{d^2f}{dp^2}(p, x_i^T) \cdot (f(p, x_i^T) - y_i) \right)$$
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Summary and Next Steps
For \( y = f(p, x) = p^T \cdot x = x^T \cdot p \)

\[
E(p, X, y) = \sum_{i=0}^{m-1} (f(p, x_i^T) - y_i)^2 = \sum_{i=0}^{m-1} (x_i \cdot p - y_i)^2
\]

and hence

\[
\frac{dE}{dp}(p, X, y) = 2 \cdot \sum_{i=0}^{m-1} (x_i \cdot p - y_i) \cdot x_i \in \mathbb{R}^{1 \times n}
\]

and

\[
\frac{d^2E}{dp^2}(p, X, y) = 2 \cdot \sum_{i=0}^{m-1} x_i^T \cdot x_i \in \mathbb{R}^{n \times n}.
\]
We present implementations for the solution of the convex unconstrained minimization problem

$$\min_{p \in \mathbb{R}^n} E(p, X, y)$$

using

- steepest descent
- Newton’s method.

See source code.
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Summary and Next Steps
For \( y = f(p, x) = (p^T \cdot x)^2 = (x^T \cdot p)^2 \)

\[
E(p, X, y) = \sum_{i=0}^{m-1} (f(p, x_i^T) - y_i)^2 = \sum_{i=0}^{m-1} (x_i \cdot p - y_i)^2
\]

and hence

\[
\frac{dE}{dp}(p, X, y) = 4 \cdot \sum_{i=0}^{m-1} ((x_i \cdot p)^3 - y_i \cdot x_i \cdot p) \cdot x_i
\]

and

\[
\frac{d^2E}{dp^2}(p, X, y) = 4 \cdot \sum_{i=0}^{m-1} (3 \cdot (x_i \cdot p)^2 - y_i) \cdot x_i^T \cdot x_i.
\]
We present implementations for the solution of the convex unconstrained minimization problem

$$\min_{p \in \mathbb{R}^n} E(p, X, y)$$

using

- steepest descent
- Newton’s method with differentiation performed
  - symbolically
  - approximately (finite differences)
  - algorithmically (dco/c++)

See source code.
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Summary

- Introduction to sample code approaching the Modern Family example as a general convex minimization problem.
- Solution of Modern Family problem using steepest descent and Newton’s method.

Next Steps

- Run the sample code.
- Compare the results.
- Continue the course to find out more …